DOI: 10.36108/ujees/2202.40.0230

UNIOSUN Journal of Engineering and Environmental Sciences. Vol. 4 No. 2. Sept. 2022

Development of an Order (k+3) Block-Hybrid Linear Multistep Method for the Direct Solution of General Second Order Initial Value Problems

Muritala, F., Kolawole, M.K., Oyedeji A.A., Lawal, J.O., Alaje A.I.

Abstract: Block hybrid linear multistep method was proposed to overcome the Dahl Quist order barrier for linear multistep methods. This research aims to answer questions relating to the convergence, accuracy, and effectiveness of the block hybrid method when utilized to obtain the solution of Initial Value Problems (IVPs). In this research, an order (k+3) block hybrid method applicable to obtain the direct solution of IVP's of ordinary differential equations (ODEs) is presented. Collocation and interpolation of power series at finely selected grid points were used to improve the method's consistency, convergence, accuracy and zero stability. Linear problems were solved to show the accuracy and efficiency of the proposed method, and the error obtained from the comparison of exact and approximate results shows that the proposed method is effective in solving the class of problem.

Keywords: Block-Hybrid, Collocation, LMM, Power series, Second Order IVPs.

I. Introduction

In this research work, we consider an approximate solution of general second-order initial value problems (IVPs) of the form

$$\left.\begin{array}{l} \ddot{y}(x) = f[x, y(x), \dot{y}(x)] \\ \text{subject to} \\ y(x_0) = y_0, \dot{y}(x_0) = \dot{y}_0 \end{array}\right\} \qquad (1)$$

f is continuously differentiable on the given interval $[a, b]$. An ordinary differential equation is an equation which has all its dependent variables and its derivatives as a function of its independent variable [1]. Equation (1) has a wide range of applications because many problems that are encountered in sciences, real-life, control theory, and engineering are modelled into Differential Equations. This is why the numerical solution of (1) is of great interest to researchers such as [2].

Muritala, F., Oyedeji A.A. and Lawal, J.O. (Department of Mathematics, Federal University of Technology Minna, Niger State)

Kolawole, M.K. and Alaje A.I. (Department of Mathematical Sciences, Osun State University, Osogbo , Osun State)

Corresponding author: mutairu.kolawole@uniosun.edu.ng

Phone Number: +2348167756777

Submitted: 28-01-2022

Accepted: 27-06-2022

Since only few of these problems can be solved analytically, there is need to study numerical methods capable of handling the problems such as the one applied by [3].

Conventionally, equation (1) is reducible to a first-order ODE and appropriate numerical methods such as the Euler method can be used to solve the resultant system [4]. The reduction process and the setbacks of this approach have been discussed by numerous authors among them is [5]. To speed up computation, achieve better accuracy, reduce computational time and eliminate overlapping of solution model, Block methods for approximating the numerical solution of equation (1) have been vastly explored in literature [6]. According to [7] and [8], block-hybrid methods were first presented to overcome zero stability barriers that happened in block methods indicated in [9], while hybrid methods were first introduced to overcome zero stability barriers that occurred in block methods mentioned in [8]. The method of interpolation and collocation of the power series approximation to generate a continuous Linear Multi-Step Method (LMM) has been adopted by many scholars [10]. For the direct solution of second-order ordinary differential equation, a single-step hybrid-block technique of order five was developed by researcher such as [11]. In this research, a block hybrid approach of order (k+3) suitable for direct computation of the solution of second-order IVP's was developed. The developed scheme was extended to two presented problems and the convergence of the approximate solution to the exact solution shows that the method is elegant.

II. Materials and Methods

A. Derivation of the Method

As an approximate solution of equation (1), we assumed a power series of a single variable.

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j \qquad (2)$$

Where a_j are the real unknown parameters to be determined and $r+s$ is the sum of the number of interpolation and number of collocation points? The first and second derivatives of (2) are given as;

$$\dot{y}(x) = \sum_{j=0}^{r+s-1} j a_j x^{j-1} \qquad (3)$$

$$\ddot{y}(x) = \sum_{j=1}^{r+s-1} j(j-1) a_j x^{j-2} \qquad (4)$$

The comparison of (4) and (1) gives rise to

$$f(x, y, \dot{y}) = \sum_{j=2}^{r+s-1} j(j-1) a_j x^{j-2} \qquad (5)$$

Equation (2) is collocated at x_{n+j} for $j = 0, \frac{1}{10}$ and (5) is as well collocated at x_{n+j}, $j = 0, \frac{1}{10}, 1, \frac{11}{10}, 2$.. This gives rise to a system of nonlinear equations

$$Ax = b \qquad (6)$$

Where the value of A is given below

$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix}, \; x = \left.\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_n + \frac{h}{10} & \left(x_n + \frac{h}{10}\right)^2 & \left(x_n + \frac{h}{10}\right)^3 & \left(x_n + \frac{h}{10}\right)^4 & \left(x_n + \frac{h}{10}\right)^5 & \left(x_n + \frac{h}{10}\right)^6 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 \\ 0 & 0 & 2 & 6\left(x_n + \frac{h}{10}\right) & 12\left(x_n + \frac{h}{10}\right)^2 & 20\left(x_n + \frac{h}{10}\right)^3 & 30\left(x_n + \frac{h}{10}\right)^4 \\ 0 & 0 & 2 & 6(x_n + h) & 12(x_n + h)^2 & 12(x_n + h)^{\backslash 3} & 12(x_n + h)^4 \\ 0 & 0 & 2 & 6\left(x_n + \frac{11h}{10}\right) & 12\left(x_n + \frac{11h}{10}\right)^2 & 20\left(x_n + \frac{11h}{10}\right)^3 & 30\left(x_n + \frac{11h}{10}\right)^4 \\ 0 & 0 & 2 & 6(x_n + 2h) & 12(x_n + 2h)^2 & 20(x_n + 2h)^3 & 30(x_n + 2h)^4 \end{bmatrix}\right\} \; and \; b = \begin{bmatrix} y_n \\ \frac{y_{n+1}}{10} \\ f_n \\ \frac{f_{n+1}}{10} \\ f_{n+1} \\ \frac{f_{n+11}}{10} \\ f_{n+2} \end{bmatrix} \qquad (6)$$

Computing $\alpha_j, j = 0(1)6$ in equation (6) above using the method of matrix inversion and then substituting into the proposed formulae from equation (2) gives the continuous formulae expressed as follows.

$$y(x) = \left\{ \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^2 \left(\sum_{j=0}^{k} \beta_j(x) f_{n+j} + \beta_v(x) f_{n+v} \right) \right\} \qquad (7)$$

Where the approximate solution of the IVP is denoted by y(x), α_j *and* β_j are the coefficients that are continuously differentiable. Since (7) is continuous and differentiable, then α_0 and β_0

are not both zero. The block method is presented as a single r-point multistep method of the form;

$$\begin{Bmatrix} y_m = \left[y_{n+1}, y_{n+2}, \ldots y_{n+r}\right]^T + y_{m-1} = \left[y_{n-1}, y_{n-2}, \ldots y_{n-r}\right]^T \\ F_m = \left[f_{n+1}, f_{n+2}, \ldots, f_{n+k}\right]^T, y_{m-1} = \left[f_{n-1}, f_{n-2}, \ldots f_{n-k}\right]^T \end{Bmatrix} \quad (8)$$

The obtained coefficients of y_{n+j} and f_{n+j}, are substituted into the continuous scheme in equation (7) and are in turn evaluated using non-interpolating points that are $x_n, x_{n+\frac{1}{10}}, x_{n+1}, x_{n+\frac{11}{10}}, x_{n+2.}$ yields the following derivative scheme presented in equation (9) to (11);

$$y_{n+1} = \left\langle \begin{array}{l} -\frac{105477}{440000} f_n h^2 + \frac{30671}{60000} f_{n+1} h^2 \\ +\frac{3167}{760000} f_{n+2} h^2 + \frac{62281}{114000} f_{n+\frac{1}{10}} h^2 \\ -\frac{9183}{22000} f_{n+\frac{11}{10}} h^2 - 9y_n + 10y_{n+\frac{1}{10}} \end{array} \right\rangle \quad (9)$$

$$y_{n+\frac{11}{10}} = \left\langle \begin{array}{l} -\frac{34139}{120000} f_n h^2 + \frac{117997}{180000} f_{n+1} h^2 \\ +\frac{34837}{6840000} f_{n+2} h^2 + \frac{54461}{85500} f_{n+\frac{1}{10}} h^2 \\ -\frac{521}{1125} f_{n+\frac{11}{10}} h^2 - 10y_n + 11y_{n+\frac{1}{10}} \end{array} \right\rangle \quad (10)$$

$$y_{n+\frac{11}{10}} = \left\langle \begin{array}{l} -\frac{561431}{660000} f_n h^2 + \frac{67013}{540000} f_{n+1} h^2 \\ +\frac{11237}{180000} f_{n+2} h^2 + \frac{15097}{9000} f_{n+\frac{1}{10}} h^2 \\ -\frac{26353}{99000} f_{n+\frac{11}{10}} h^2 - 19y_n + 20y_{n+\frac{1}{10}} \end{array} \right\rangle \quad (11)$$

The first derivative is obtained by differentiating the continuous scheme in equation (7) with respect to x and the results are evaluated at all interpolation and collocation points $x_n, x_{n+\frac{1}{10}}$ and $x_n, x_{n+\frac{1}{10}}, x_{n+1}, x_{n+\frac{11}{10}}, x_{n+2.}$ **respectively** which give rise to equations (12-16);

$$hz_n = \left\langle \begin{array}{l} -\frac{41431}{1320000} f_n h^2 + \frac{1039}{540000} f_{n+1} h^2 \\ +\frac{509}{20520000} f_{n+2} h^2 - \frac{19471}{1026000} f_{n+\frac{1}{10}} h^2 \\ -\frac{941}{594000} f_{n+\frac{11}{10}} h^2 - 10y_n + 10y_{n+\frac{1}{10}} \end{array} \right\rangle \quad (12)$$

$$hz_{n+\frac{1}{10}} = \left\langle \begin{array}{l} \frac{19471}{1320000} f_n h^2 - \frac{1009}{540000} f_{n+1} h^2 \\ -\frac{163}{6840000} f_{n+2} h^2 + \frac{9133}{256500} f_{n+\frac{1}{10}} h^2 \\ +\frac{19}{12375} f_{n+\frac{11}{10}} h^2 - 10y_n + 10y_{n+\frac{1}{10}} \end{array} \right\rangle \quad (13)$$

$$hz_{n+1} = \left\langle \begin{array}{l} -\frac{591431}{1320000} f_n h^2 + \frac{761039}{540000} f_{n+1} h^2 \\ +\frac{63503}{6840000} f_{n+2} h^2 + \frac{930529}{1026000} f_{n+\frac{1}{10}} h^2 \\ -\frac{183647}{198000} f_{n+\frac{11}{10}} h^2 - 10y_n + 10y_{n+\frac{1}{10}} \end{array} \right\rangle \quad (14)$$

$$hz_{n+\frac{11}{10}} = \left\langle \begin{array}{l} -\frac{196843}{440000} f_n h^2 + \frac{262997}{180000} f_{n+1} h^2 \\ +\frac{189511}{20520000} f_{n+2} h^2 + \frac{77461}{85500} f_{n+\frac{1}{10}} h^2 \\ -\frac{65261}{74250} f_{n+\frac{11}{10}} h^2 - 10y_n + 10y_{n+\frac{1}{10}} \end{array} \right\rangle \quad (15)$$

$$hz_{n+2} = \left\langle \begin{array}{l} -\frac{36407}{40000} f_n h^2 - \frac{292987}{180000} f_{n+1} h^2 \\ +\frac{5240509}{20520000} f_{n+2} h^2 + \frac{526843}{342000} f_{n+\frac{1}{10}} h^2 \\ +\frac{145369}{54000} f_{n+\frac{11}{10}} h^2 - 10y_n + 10y_{n+\frac{1}{10}} \end{array} \right\rangle \quad (16)$$

And the proposed Block-Hybrid method is given as:

$$y_{n+1} = \left\langle \begin{array}{l} -\frac{5}{24} f_n h^2 + \frac{55}{108} f_{n+1} h^2 + \frac{17}{4104} f_{n+2} h^2 \\ +\frac{290}{513} f_{n+\frac{1}{10}} h^2 - \frac{10}{27} f_{n+\frac{11}{10}} h^2 + y_n + z_n h \end{array} \right\rangle \quad (17)$$

$$y_{n+2} = \left\langle \begin{array}{l} -\frac{26}{33} f_n h^2 + \frac{20}{27} f_{n+1} h^2 + \\ \frac{32}{513} f_{n+2} h^2 + \frac{880}{513} f_{n+\frac{1}{10}} h^2 \\ +\frac{80}{297} f_{n+\frac{11}{10}} h^2 + y_n + 2 z_n h \end{array} \right\rangle \quad (18)$$

$$y_{n+\frac{1}{10}} = \left\langle \begin{array}{l} -\frac{41431}{13200000} f_n h^2 - \frac{1039}{5400000} f_{n+1} h^2 \\ -\frac{509}{205200000} f_{n+2} h^2 + \frac{19471}{10260000} f_{n+\frac{1}{10}} h^2 \\ +\frac{941}{5940000} f_{n+\frac{11}{10}} h^2 + \frac{1}{10} z_n h + y_n \end{array} \right\rangle \quad (19)$$

$$y_{n+\frac{11}{10}} = \left\langle \begin{array}{l} -\frac{299959}{1200000} f_n h^2 + \frac{3528481}{5400000} f_{n+1} h^2 \\ +\frac{1039511}{205200000} f_{n+2} h^2 + \frac{6749501}{10260000} f_{n+\frac{1}{10}} h^2 \\ -\frac{249139}{540000} f_{n+\frac{11}{10}} h^2 + \frac{11}{10} z_n h + y_n . \end{array} \right\rangle \quad (20)$$

$$z_{n+1} = \left\langle \begin{array}{l} -\frac{5}{12} f_n h + \frac{38}{27} f_{n+1} h + \frac{1}{108} f_{n+2} h \\ +\frac{25}{27} f_{n+\frac{1}{10}} h - \frac{25}{27} f_{n+\frac{11}{10}} h + z_n \end{array} \right\rangle \quad (21)$$

$$z_{n+2} = \left\langle \begin{array}{l} -\frac{29}{33} f_n h - \frac{44}{27} f_{n+1} h \\ +\frac{131}{513} f_{n+2} h + \frac{800}{513} f_{n+\frac{1}{10}} h \\ +\frac{800}{297} f_{n+\frac{11}{10}} h + z_n \end{array} \right\rangle \quad (22)$$

$$z_{n+\frac{1}{10}} = \left\langle \begin{array}{l} \frac{30451}{660000} f_n h - \frac{64}{16875} f_{n+1} h \\ -\frac{499}{10260000} f_{n+2} h + \frac{56003}{1026000} f_{n+\frac{1}{10}} h \\ +\frac{1853}{594000} f_{n+\frac{11}{10}} h + z_n \end{array} \right\rangle \quad (23)$$

$$z_{n+\frac{11}{10}} = \left\langle \begin{array}{l} -\frac{24959}{60000} f_n h + \frac{49247}{33750} f_{n+1} h \\ +\frac{94501}{10260000} f_{n+2} h + \frac{949003}{1026000} f_{n+\frac{1}{10}} h \\ -\frac{47377}{54000} h f_{n+\frac{11}{10}} h + z_n \end{array} \right\rangle \quad (24)$$

B. The Block's Analysis

i. The Block's Order and Error Constant

Let ℓ be a linear difference operator on the method defined as:

$$\ell[y(x);h] = \sum_{i=0}^{k-1} \left[\alpha_j y(x+ih) - h^2 \beta_j y''(x+ih)\right]$$

Where $y(x)$ is a continuously differentiable function on the interval $[\alpha, \beta]$ Expanding (22) using Taylor's series about y(x) and collecting the coefficients of power of h yields;

$$\ell[y(x);h] = \left\langle \begin{array}{l} c_0 y(x) + c_1 h \dot{y}(x) \\ + c_1 h^2 \dot{y}(x)(x) + \cdots + \\ c_q h^q y^q(x) + 0(h^{q+1}) \end{array} \right\rangle \quad (25)$$

whose coefficients $c_q \;\; \forall \; q = 0,1,2,\cdots$ defined as:

$$c_0 = \sum_{i=0}^{k} \left[\alpha_i = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k\right] \quad (26)$$

$$c_1 = \sum_{i=0}^{k} \left\{ \begin{bmatrix} i\alpha_i - \beta_i - = (\alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots \\ + k\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k) \end{bmatrix} \right\} \quad (27)$$

$$c_2 = \sum_{i=0}^{k} \frac{i^2}{2!} \alpha_i - \sum_{i=0}^{k} \beta_i$$

Such that

$$c_2 = \left\langle \begin{array}{l} \frac{1}{2!}(\alpha_1 + 2^2 \alpha_2 + 3^2 \alpha_3 + \cdots + k^2 \alpha_k) \\ -(\beta_0 + 2\beta_1 + 3\beta_2 + \cdots + k\beta_k) \end{array} \right\rangle \quad (28)$$

$\vdots \qquad\qquad \vdots$

$$c_q = \sum_{i=0}^{k} \left\{ \frac{1}{q!} i^q \alpha_i - \frac{1}{(q-2)!} i^{q-2} \beta_i \right\} \quad (29)$$

$$c_q = \left\langle \begin{array}{l} \frac{1}{q!}(\alpha_1 + 2^q \alpha_2 + 3^q \alpha_3 + k^q \alpha_k \\ -\frac{1}{(q-2)!}(\beta_1 + 2^{(q-2)} \beta_2 + 3^{(q-2)} \beta_3 \\ + \cdots + k^{(q-2)} \beta_k \end{array} \right\rangle (30)$$

Thus the block schemes are of order p if $C_0 = C_1 = C_2 = \cdots = C_{P+1} = 0$ and C_{P+2} is not equal to zero, therefore C_{P+2} called the error constant. Thus, the error due to truncation at the local level is $T_{n+k} = c_{p+2} h^{p+2} y^{p+2}(x) + 0(h^{p+3})$. The order of the block is p=5 and the error constants obtained by comparing coefficients of h is:

$$c_{p+2} = \left[\frac{144601}{800000000}, \frac{262823}{1200000000}, \frac{1233803}{1200000000} \right]^T$$

ii. Consistency

A linear Multistep Method (LMM) is said to be consistent if order $p \geq 1$ and the following axioms are satisfied;

i. $\sum_{i=0}^{k} a_i = 0$

ii. $p(1) = \dot{p}(1) = 0$

iii. $\ddot{p} = 2!\sigma(r) = 0$

Where $p(r) = \sum_{i=0}^{k} a_j r^j$ and $\sigma(r) = \sum_{i=0}^{k} \beta_j r^j$ are the characteristics polynomial of the first and second order of the proposed method respectively. Making reference to [12], an associated block method is consistent if $p \geq 1$, since the proposed method is of order p=5, it is concluded that the method is consistent.

iii. Zero Stability

Let $A^{(0)} y_m = \sum_{i=1}^{k} \alpha^i y_{m-1} + h^2 \left(\sum_{i=0}^{k} \beta^i f_{m-i} \right)$ (26)

Be a single block r-point multistep technique. Applying the block in equation (21) we have;

$$\det\left[\Omega I - A_1^{(1)}\right]$$

$$= \left\| \begin{bmatrix} \Omega & 0 & 0 & 0 \\ 0 & \Omega & 0 & 0 \\ 0 & 0 & \Omega & 0 \\ 0 & 0 & 0 & \Omega \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right\| = 0$$

$$\Omega^3(\Omega - 1) = 0 \rightarrow \Omega_1 = 0, \Omega_2 = 0, \Omega_3 = 0, \Omega_4 = 1.$$

Since there exist no roots greater than $|\Omega| = 1$ this implies the zero-stability of the derived Block Hybrid Method.

iv. Convergence

Fatunla (1973) stated that consistency and zero stability are the necessary and sufficient conditions required for the linear multistep method (LMM) to be convergent.

C. Implementation of the Method

The performance of the method is tested on some linear real-life problems and a system of equations of second-order initial value problems. The results obtained were compared with the exact solution to determine the absolute errors.

i. Application

In this section, the potency and the accuracy of the proposed numerical scheme are validated by applying the method to solve different problems resulting from areas of application. The obtained solution is compared with the exact solution to know the absolute error.

ii. Cooling Problem

The ODE governing the reduction in temperature $y(x)$ of a body placed in a certain room after some period x is given as

$$3\ddot{y}(x) + \dot{y}(x) = 0.$$

Subject to the conditions

$y(0) = 60$, $y(6\log_e 4) = 35$

The exact solution to the problem is

$$y(x)=\frac{80}{3}e^{-\frac{1}{3}x}+\frac{100}{3}$$

The governing ODE is restructured to suit the computation of the proposed scheme such that

$$\ddot{y}(x)=-\frac{\dot{y}(x)}{3},$$

Subject to $y(0)=60,\ y'(0)=\frac{80}{9}$

$h=0.1$ Is chosen as step size and the results obtained at various points of h are presented on table1.

iii. Slightly Stiff linear problem

$$\ddot{y}(x)=\dot{y}(x),$$

With the initial condition

$$y(0)=0,\ y'=-1, h=0.1 \quad (28)$$

The exact solution is

$$y(x)=1-e^{x}$$

The obtained results at various points of h are similarly presented in Table 2.

iv. Maple 18 Algorithm for problem 1

The following algorithm is applicable in generating the approximate results of problem 1 presented in Table 1.

```
Digits: = 30: N: = 10: h := 1/N: x [0] := 0: y [0] := 60: z [0] := -80/9:
A := y[n+1] = -(5/24)*f[n]*h^2+(55/108)*f[n+1]*h^2+(17/4104)*f[n+2]*h^2+(290/513)*f[n+1/10]*h^2-(10/27)*f[n+11/10]*h^2+y[n]+z[n]*h, y[n+2] = -(26/33)*f[n]*h^2+(20/27)*f[n+1]*h^2+(32/513)*f[n+2]*h^2+(880/513)*f[n+1/10]*h^2+(80/297)*f[n+11/10]*h^2+y[n]+2*z[n]*h, y[n+1/10] = (41431/13200000)*f[n]*h^2-(1039/5400000)*f[n+1]*h^2-(509/205200000)*f[n+2]*h^2+(19471/10260000)*f[n+1/10]*h^2+(941/5940000)*f[n+11/10]*h^2+(1/10)*z[n]*h+y[n], y[n+11/10] = -(299959/1200000)*f[n]*h^2+(3528481/5400000)*f[n+1]*h^2+(1039511/205200000)*f[n+2]*h^2+(6749501/10260000)*f[n+1/10]*h^2-(249139/540000)*f[n+11/10]*h^2+y[n]+(11/10)*z[n]*h, z[n+1] = -(5/12)*h*f[n]+(38/27)*h*f[n+1]+(1/108)*h*f[n+2]+(25/27)*h*f[n+1/10]-(25/27)*h*f[n+11/10]+z[n], z[n+2] = -(29/33)*h*f[n]-(44/27)*h*f[n+1]+(131/513)*h*f[n+2]+(800/513)*h*f[n+1/10]+(800/297)*h*f[n+11/10]+z[n], z[n+1/10] = (30451/660000)*h*f[n]-(64/16875)*h*f[n+1]-(499/10260000)*h*f[n+2]+(56003/1026000)*h*f[n+1/10]+(1853/594000)*h*f[n+11/10]+z[n], z[n+11/10] = -(24959/60000)*h*f[n]+(49247/33750)*h*f[n+1]+(94501/10260000)*h*f[n+2]+(949003/1026000)*h*f[n+1/10]-(47377/54000)*h*f[n+11/10]+z[n]:
for n from 0 by 2 to N do f[n] := -(1/3)*z[n]; f[n+1/10] := -(1/3)*z[n+1/10]; f[n+10*(1/10)] := -(1/3)*z[n+10*(1/10)]; f[n+11/10] := -(1/3)*z[n+11/10]; f[n+2] := -(1/3)*z[n+2]; P := fsolve({A}); Q := eval([y[n+1], y[n+2], y[n+1/10], y[n+11/10], z[n+1], z[n+2], z[n+1/10], z[n+11/10]], P); y[n+1] := Q[1]; y[n+2] := Q[2]; y[n+1/10] := Q[3]; y[n+11/10] := Q[4]; z[n+1] := Q[5]; z[n+2] := Q[6]; z[n+1/10] := Q[7]; z[n+11/10] := Q[8] end do:
> for j from 0 to N do y[j] := evalf[20](y[j]) end do;
```

v. Maple 18 Algorithm for Problem 2.

Similarly, the results of problem 2 presented in Table 2 is obtained using the following algorithm.

```
> Digits: = 30: N: = 10: h := 1/N: x[0] := 0: y[0] := 0: z[0] := -1: A := y[n+1] = -(5/24)*f[n]*h^2+(55/108)*f[n+1]*h^2+(17/4104)*f[n+2]*h^2+(290/513)*f[n+1/10]*h^2-(10/27)*f[n+11/10]*h^2+y[n]+z[n]*h, y[n+2] = -(26/33)*f[n]*h^2+(20/27)*f[n+1]*h^2+(32/513)*f[n+2]*h^2+(880/513)*f[n+1/10]*h^2+(80
```

/297)*f[n+11/10]*h^2+y[n]+2*z[n]*h, y[n+1/10] = (41431/13200000)*f[n]*h^2-(1039/5400000)*f[n+1]*h^2-(509/205200000)*f[n+2]*h^2+(19471/10260000)*f[n+1/10]*h^2+(941/5940000)*f[n+11/10]*h^2+(1/10)*z[n]*h+y[n], y[n+11/10] = -(299959/1200000)*f[n]*h^2+(3528481/5400000)*f[n+1]*h^2+(1039511/205200000)*f[n+2]*h^2+(6749501/10260000)*f[n+1/10]*h^2-(249139/540000)*f[n+11/10]*h^2+y[n]+(11/10)*z[n]*h, z[n+1] = -(5/12)*h*f[n]+(38/27)*h*f[n+1]+(1/108)*h*f[n+2]+(25/27)*h*f[n+1/10]-(25/27)*h*f[n+11/10]+z[n], z[n+2] = -(29/33)*h*f[n]-(44/27)*h*f[n+1]+(131/513)*h*f[n+2]+(800/513)*h*f[n+1/10]+(800/297)*h*f[n+11/10]+z[n], z[n+1/10] = (30451/660000)*h*f[n]-(64/16875)*h*f[n+1]-(499/10260000)*h*f[n+2]+(56003/1026000)*h*f[n+1/10]+(1853/594000)*h*f[n+11/10]+z[n], z[n+11/10] = -(24959/60000)*h*f[n]+(49247/33750)*h*f[n+1]+(94501/10260000)*h*f[n+2]+(949003/1026000)*h*f[n+1/10]-(47377/54000)*h*f[n+11/10]+z[n]:

for n from 0 by 2 to N do f[n] := z[n]; f[n+1/10] := z[n+1/10]; f[n+10*(1/10)] := z[n+10*(1/10)]; f[n+11/10] := z[n+11/10]; f[n+2] := z[n+2]; P := fsolve({A}); Q := eval([y[n+1], y[n+2], y[n+1/10], y[n+11/10], z[n+1], z[n+2], z[n+1/10], z[n+11/10]], P); y[n+1] := Q[1]; y[n+2] := Q[2]; y[n+1/10] := Q[3]; y[n+11/10] := Q[4]; z[n+1] := Q[5]; z[n+2] := Q[6]; z[n+1/10] := Q[7]; z[n+11/10] := Q[8] end do;

> for j from 0 to N do y[j] := evalf[20](y[j]) end do;

III. Results and Discussion

The results obtained from the proposed method with step number two and accuracy order five were compared with the exact solution and other methods. The accuracy of the developed method was examined using two different problems and their corresponding results are discussed below

In Table 1, the absolute error signifies that the approximate solution accelerates and converges to the exact solution with minimal error.

It was observed from table 2 that the greatest error recorded from the proposed method is 7.76587 E-11 which shows that the accuracy of the proposed method is to a great extent.

Table 1: Numerical Results of Problem 1

X	Exact Results	Computed Results	Absolute Error
0	60	60	0
0.1	59.125762679520157388	59.125762679519946596	2.1079E-13
0.2	58.280186267509806339	58.280186267508610635	1.1957E-12
0.3	57.462331147625588618	57.462331147622161666	3.4269E-12
0.4	56.671288507811932107	56.671288507805616398	6.3157E-12
0.5	55.906179330416375308	55.906179330406069385	1.0305E-11
0.6	55.166153415412849564	55.166153415398000705	1.4848E-11
0.7	54.450388435647511050	54.450388435627149172	2.0361E-11
0.8	53.758089023057298472	53.758089023030964860	2.6333E-11
0.9	53.088485884845809762	53.088485884812653343	3.3156E-11

Table 2: Numerical Results of Problem 2

X	Exact Results	Computed Results	Absolute Error
0.1	0.10517091807564762481	0.10517091809570308609	1.57030 E-11
0.2	0.22140275816016983392	0.22140275827765879433	7.76587 E-11
0.3	0.34985880757600310398	0.34985880793383639184	3.57833 E-10
0.4	0.49182469764127031782	0.49182469835665867039	7.15388 E-10
0.5	0.64872127070012814684	0.64872127197271171632	1.27258 E-09
0.6	0.82211880039050897487	0.82211880239117632891	2.00066 E-09
0.7	1.01375270747047652162	1.01375271047371182210	3.00323 E-09
0.8	1.22554092849246760457	1.22554093274086051460	4.24839 E-09
0.9	1.45960311115694966380	1.45960311702318349860	5.86623 E-09
1	1.71828182845904523536	1.71828183628078357650	7.82173 E-09

IV. Conclusion

In this study, an implicit block-hybrid method for solving ODE of second order was proposed. The consistency, convergence, and zero-stability of the method were studied and the newly developed scheme offers an opportunity to record an accurate approximate solution at acceptable sites when applied to the class of problem solved. It was seen from the presented tables generated with the algorithm coded with Maple 18 software that the method produced a better approximation when compared with the exact solution as the error converges to digits near zero.

References

[1] Muhammad, R. and Yahaya, Y.A. "A Sixth Order Implicit Hybrid Backward Differentiation Formulae (HBDF) for Block Solution of Ordinary Differential Equations", *American Journal of Mathematics and Statistics,* vol. 2, no 4, 2012, pp. 89-94.

[2] Skwame, Y., Sabo, J. and Mathew, M. "The Treatment of Second Order Ordinary Differential Equations Using Equidistant One-Step Block Hybrid", *Asian Journal of Probability and Statistics,* vol. 5, no 3, 2019, pp. 1-9.

[3] Hussain, K., Ismail, F. and Senu, N. "Solving Directly Special Fourth-order Ordinary Differential Equations using Runge-Kutta Type Method", *Journal of Computer and Applied Mathematics,* vol. 306, 2016, pp. 179-199.

[4] Dahlquist, G. "Some Properties of Linear Multistep Methods and One-leg Methods for Ordinary Differential Equations", Report TRITA-NA-7+04, Department of Computer Science, Royal Institute of Technology, Stockholm, 1979.

[5] Henrici, P. "Discrete Variable Methods in Ordinary Differential Equations*", John Wiley & Sons*, New York, 1962.

[6] Adeniran, A.O. and Ogundare, B.S. "An Efficient Hybrid Numerical Scheme for Solving General Second Order Initial Value Problems (IVPs*)", International Journal of Applied Mathematical Research*, vol. 4, no 2, 2015, pp. 411-419.

[7] Lambert, J.D. "Computational Methods in Ordinary Differential Equations", *John Wiley and Sons*, New York, 1973.

[8] Yahaya, Y.A. and Badmus, A.M. "A Class of Collocation Methods for General Second-order Ordinary Differential Equations", *African Journal of Mathematics & Computer Science Research,* vol. 2, no 6, 2009, pp. 69-72.

[9] Omole, E.O. and Ogunware, B.G. "3-Point Single Hybrid Block Method (3PSHBM) for Direct Solution of General Second Order Initial Value Problem of Ordinary Differential Equations", *Journal of Scientific Research and Reports,* vol. 20, no 3, 2018, pp. 1-11.

[10] Anake, A.T., Awoyemi, D.O. and

Adesanya, A.A. “A One-Step Method for the Solution of General Second Order Ordinary Differential Equations”, *International Journal of Science and Technology*, vol. 2, no 4, 2012.

[11] Abdelrahim, R.F. and Omar, Z. “Direct Solution of the Second-order Ordinary Differential Equation using a Single-step Hybrid Block Method of Order Five”, *Mathematical and Computational Applications,* vol. 21, 2016, pp. 1-12