SIX-STEP SECOND DERIVATIVE NON-HYBRID BLOCK METHOD BACKWARD DIFFERENTIATION FORMULA FOR STIFF SYSTEMS

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Abstract
This paper presents an advanced six-step linear multistep method of order five, based on second derivative non-hybrid block backward differentiation formula (BDF), developed for numerical solution of stiff systems of ordinary differential equations. The development of the multistep collocation approach is carried out using the matrix inversion techniques. The block structure facilitates simultaneous computation of multiple solution points, improving computational efficiency. Moreover, the power series is adopted as basis function for driving the discrete and continuous formulations. The analysis of the method such as consistency, zero-stability, order and error constants is presented, confirming its suitability for stiff systems. Numerical experiments on standard are tested on stiff and non-stiff ordinary differential equations showed the new method outperforms existing method in terms of accuracy.

Keywords
Block Method, Non-Hybrid, Second Derivative, Backward Differentiation Formula, Stiff ODEs

1. INTRODUCTION

Mathematical statement containing one or more derivative of physical phenomena in sciences, social sciences and engineering often lead to initial value problem (Mohammed and Adeniyi, 2013). In some cases, the differential equations could be solved analytically while others may be too complicated to use analytic methods of solutions. Numerical methods are often used to give approximate solutions to these differential equations. In view of the importance of differential equations, Stiff problems are important because they occur in various applications of science. It arises when modeling chemical reaction, Reaction-diffusion systems, electrical circuits, mechanics, meteorology, oceanography and vibrations (Idrees *et al.*, 2013). A hybrid block scheme is a numerical method for solving ordinary differential equations that combines both implicit and explicit integration schemes. It is designed to handle stiff systems, where the solution shows rapid changes over different time scales. The hybrid block scheme divides the time intervals into smaller sub intervals or blocks. Within each block, an implicit method is used to handle the stiff components of the system, while an explicit method is used to handle the non-stiff components. The implicit method provides stability and accuracy for stiff part, while explicit method allows for efficient computation of the non-stiff part (Hairer *et al.*, 1993). One way to deal with stiff problems is to use stable implicit methods. The most popular methods for the solution of stiff initial value problems for ordinary differential equations are the Backward Differentiation Formula (BDFs). Others are Power series method (Tahmasbi and Fard, 2008), block method (Ibijola *et al.*, 2011), hybrid block methods (Kumleng *et al.*, 2013). However, there are few schemes developed for solving stiff ODEs using block methods. In this research work, we will adopt the idea of interpolation and collocation which Lie and Norselt (1989), Onumanyi *et al.* (1994), (1999) and Sunday (2022) referred to as the multistep collocation method (MC). The scheme presented below following Sunday (2022).

2. METHODOLOGY

A general form of the continuous of the $k-step$ linear multistep method as defined by Sunday (2022) is given in equation 1 as:

$$y(x)=\sum_{j=0}^{t-1}\alpha_j(x)y_{n+j}+h\sum_{j=0}^{m-1}\beta_j(x)f_{n+j} \tag{1}$$

and its extension it to the second derivative gives the general form of the continuous k –step 2nd derivative linear multistep method as shown in equation 2:

$$y(x) = \sum_{j=0}^{r-1} \alpha_j(x) y_{n+j} + h \sum_{j=0}^{t-1} \beta_j(x) f_{n+j} + h^2 \sum_{j=0}^{m-1} \gamma_j(x) \quad (2)$$

where h is the step length, s, t the number of distinct collocation points and r the number of interpolation points. Our aim is to utilize not only the interpolation point $\{x_i\}$but also several collocation point on the interpolation interval and to fit $y(x)$ for $y'(x)$ and $y''(x)$.We impose the following conditions:

$$y(x_{n+j}) = y_{n+j}, \ (j = 0.1 \ldots \ldots r-1) \quad (3)$$

$$y'(c_{n+j}) = \bar{y}'_{n+j}, \ (j = 0.1 \ldots \ldots t-1) \quad (4)$$

$$y''(c_{n+j}) = \bar{y}''_{n+j}, \ (j = 0.1 \ldots \ldots m-1) \quad (5)$$

Where $\{c_{n+j}\}$ are collocation points distributed on the step-points array,y_{n+j} is the interpolation data of $y(x)$ on x_{n+j} and $\overline{y'}_{n+j}$,$\overline{y''}_{n+j}$ are the collocation data of $y'(x)$ and $y''(x)$ respectively on$\{c_{n+j}\}$. Expressing equations (3), (4) and (5) in the matrix-vector form gives:

$$\mathrm{D}C = I \quad (6)$$

Where I is the identity matrix of dimension$(t+m) \times (t+m)$. The matrix D is defined as

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^{t+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2\overline{x_0} & \cdots & (t+m-1)\bar{x}_0^{t+m-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2\overline{x_{m-1}} & \cdots & (t+m-1)\bar{x}_{m-1}^{t+m-2} \\ 0 & 1 & 2\overline{x_{m-2}} & \cdots & (t+m-2)\bar{x}_{m-1}^{t+m-3} \end{bmatrix} \quad (7)$$

$$C = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{t-1,1} & h\beta_{0,1} & \cdots & h\beta_{m-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots & \alpha_{t-1,2} & h\beta_{0,2} & \cdots & h\beta_{m-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,t+m} & \alpha_{1,t+m} & \cdots & \alpha_{t-1,t+m} & h\beta_{0,t+m} & \cdots & h\beta_{m-1,t+m} \end{bmatrix} \quad (8)$$

The columns of $C = D^{-1}$with aids of MAPLE 18 we will obtain the continuous coefficient $\alpha_j(x)$,$\beta_j(x)$, $\gamma_j(x)$ and the continuous scheme.

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y_{n+2} + \alpha_3(x)y_{n+3} + \alpha_4(x)y_{n+4} + \alpha_5(x)y_{n+5} + h[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3} + \beta_4(x)f_{n+4} + \beta_5(x)f_{n+5} + \beta_6(x)f_{n+6}] + h^2[\gamma_0(x)f_n + \gamma_1(x)f_{n+1} + \gamma_2(x)f_{n+2} + \gamma_3(x)f_{n+3} + \gamma_4(x)f_{n+4} + \gamma_5(x)f_{n+5} + \gamma_6(x)f_{n+6}] \quad (9)$$

The matrix D of the method is expressed as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 & 64 & 128 & 256 & 512 \\ 1 & 3 & 9 & 27 & 81 & 243 & 729 & 2187 & 6561 & 19683 \\ 1 & 4 & 16 & 64 & 256 & 1024 & 4096 & 16384 & 65536 & 262144 \\ 1 & 5 & 25 & 125 & 625 & 3125 & 15625 & 78125 & 390625 & 1953125 \\ 0 & 1 & 10 & 75 & 500 & 3125 & 18750 & 109375 & 625000 & 3515625 \\ 0 & 1 & 12 & 108 & 864 & 6480 & 46656 & 326592 & 2239488 & 15116544 \\ 0 & 0 & 2 & 30 & 300 & 2500 & 18750 & 131250 & 4375000 & 5625000 \\ 0 & 0 & 2 & 36 & 432 & 4320 & 38880 & 326592 & 6612736 & 20155392 \end{bmatrix}$$

Using maple software package gives the columns of D^{-1} which are the elements of the matrix$C$. The elements of C are then used to generate the values of continuous coefficient:

$$\alpha_0(x), \alpha_1(x), \alpha_2(x), \alpha_3(x), \alpha_4(x), \alpha_5(x), \beta_0(x), \beta_1(x), \beta_2(x), \beta_3(x), \beta_4(x), \beta_5(x), \beta_6(x) \\ \gamma_0(x), \gamma_1(x), \gamma_2(x), \gamma_3(x), \gamma_4(x), \gamma_5(x) \text{ and } \gamma_6(x) \quad (10)$$

The values of the continuous coefficient (8) of the method six-step non-hybrid block scheme are given as:

$$\alpha_0(x) = 1 - \frac{2520483x}{838580h} + \frac{93566443x^2}{25157400h^2} - \frac{5730815093x^3}{2264166000h^3} + \frac{4783675087x^4}{4528332000h^4} - \frac{1280436131x^5}{4528332000h^5} + \frac{22115893x^6}{4528332000h^6} - \frac{1492499x^7}{283020750h^7} + \frac{1468003x^8}{4528332000h^8} - \frac{39259x^9}{4528332000h^9}$$

$$\alpha_1(x) = \frac{1861875x}{167716h} - \frac{14936625x^2}{670864h^2} + \frac{25522025x^3}{1341728h^3} - \frac{73052285x^4}{8050368h^4} + \frac{14230165x^5}{5366912h^5} - \frac{392080x^6}{8050368h^6} + \frac{110743x^7}{2012592h^7} - \frac{2349x^8}{670864h^8} + \frac{1549x^9}{16100736h^9}$$

$$\alpha_2(x) = -\frac{1267000x}{41929h} + \frac{3174450x^2}{41929h^2} - \frac{112479865x^3}{1509444h^3} + \frac{710196947x^4}{18113328h^4} - \frac{222923893x^5}{18113328h^5} + \frac{21619159x^6}{9056664h^6} - \frac{2528763x^7}{9056664h^7} + \frac{335615x^8}{18113328h^8} - \frac{9481x^9}{18113328h^9}$$

$$\alpha_3(x) = \frac{3263250x}{41929h} - \frac{52315825x^2}{251574h^2} + \frac{996527845x^3}{4528332h^3} - \frac{1118912771x^4}{9056664h^4} + \frac{742285745x^5}{18113328h^5} - \frac{37706863x^6}{4528332h^6} + \frac{9246763x^7}{9056664h^7} - \frac{629423x^8}{9056664h^8} + \frac{36589x^9}{18113328h^9}$$

$$\alpha_4(x) = -\frac{40937625x}{167716h} + \frac{225557775x^2}{335432h^2} - \frac{495895585x^3}{670864h^3} + \frac{1740543079x^4}{4025184h^4} - \frac{200428949x^5}{1341728h^5} + \frac{63476653x^6}{2012592h^6} - \frac{4031489x^7}{1006296h^7} + \frac{377849x^8}{1341728h^8} - \frac{33919x^9}{4025184h^9}$$

$$\alpha_5(x) = \frac{157974233x}{838580h} - \frac{8748909087x^2}{16771600h^2} + \frac{871177248187x^3}{1509444000h^3} - \frac{3079791159799x^4}{9056664000h^4} + \frac{2143523897149x^5}{18113328000h^5} - \frac{45576357197x^6}{18113328000h^6} + \frac{7283704367x^7}{2264166000h^7} - \frac{515022439x^8}{2264166000h^8} + \frac{123941911x^9}{18113328000h^9}$$

$$\beta_5(x) = -\frac{5434473x}{41929} + \frac{301080303x^2}{838580h} - \frac{9995943401x^3}{25157400h^2} + \frac{35333732477x^4}{150944400h^3} - \frac{24573318527x^5}{301888800h^4} + \frac{521604331x^6}{301888800h^5} - \frac{41562833x^7}{18868050h^6} + \frac{2927161x^8}{18868050h^7} - \frac{1401653x^9}{301888800h^8}$$

$$\beta_6(x) = \frac{800750x}{41929} - \frac{4542025x^2}{83858h} + \frac{93349765x^3}{1509444h^2} - \frac{57128221x^4}{1509444h^3} + \frac{10378891x^5}{754722h^4} - \frac{1156735x^6}{377361h^5} + \frac{621905x^7}{1509444h^6} - \frac{46309x^8}{1509444h^7} + \frac{367x^9}{377361h^8}$$

$$\gamma_5(x) = -\frac{2527110xh}{41929} + \frac{14167953x^2}{83858} - \frac{159289573x^3}{838580h} + \frac{573994441x^4}{5031480h^2} - \frac{408144931x^5}{10062960h^3} + \frac{8879021x^6}{1006296h^4} - \frac{2905711x^7}{2515740h^5} + \frac{210377x^8}{2515740h^6} - \frac{25909x^9}{10062960h^8}$$

$$\gamma_6(x) = \frac{207750xh}{41929} - \frac{1181675x^2}{83858} + \frac{1354285x^3}{838580h} - \frac{3330711x^4}{335432h^2} + \frac{1217061x^5}{335432h^3} - \frac{136545x^6}{167716h^4} + \frac{18495x^7}{167716h^5} - \frac{2779x^8}{335432h^6} + \frac{89x^9}{335432h^8}$$

Evaluating (9) at point $x = x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}$ and at $x = x_{n+5}$ and x_{n+6} and obtain the following six discrete methods which constitute the new six- step non-hybrid block method.

$$y_n - \frac{1120246727261}{187132886}y_{n+1} + \frac{1904670000}{93566443}y_{n+2} - \frac{5231582500}{93566443}y_{n+3} + \frac{16916833125}{93566443}y_{n+4} - \frac{26246727261}{187132886}y_{n+5} = \frac{30h}{93566443}[301080303f_{n+5} + 45420250f_{n+6}] + \frac{300h^2}{93566443}[41929g_n + 14167953g_{n+5} + 1181675g_{n+6}]$$

$$y_n + \frac{282639375}{121035344}y_{n+1} - \frac{149042750}{7564709}y_{n+2} + \frac{436751125}{7564709}y_{n+3} - \frac{1356078375}{7564709}y_{n+4} + \frac{16690245281}{121035344}y_{n+5} = \frac{15h}{30258836}[191425063f_{n+5} + 27567200f_{n+6}] + \frac{225h^2}{15129418}[125787g_{n+1} - 2934584g_{n+5} - 237672g_{n+6}]$$

$$y_n - \frac{1465875}{52678}y_{n+1} + \frac{24236500}{1027221}y_{n+2} + \frac{38441500}{342407}y_{n+3} - \frac{165139875}{342407}y_{n+4} - \frac{766831933}{2054442}y_{n+5} = \frac{70h}{342407}[1260137f_{n+5} + 174450f_{n+6}] + \frac{50h^2}{342407}[209645g_{n+2} + 797362g_{n+5} + 62775g_{n+6}]$$

$$y_n - \frac{4272075}{232592}y_{n+1} + \frac{3721350}{14537}y_{n+2} - \frac{5172050}{14537}y_{n+3} - \frac{7541775}{14537}y_{n+4} + \frac{147919083}{232592}y_{n+5} = \frac{15h}{58148}[1794549f_{n+5} + 263360f_{n+6}] + \frac{75h^2}{29074}[83858g_{n+3} - 85419g_{n+5} - 6728g_{n+6}]$$

$$y_n - \frac{18295875}{1187534} y_{n+1} + \frac{80561000}{593767} y_{n+2} - \frac{758102500}{593767} y_{n+3} - \frac{871880625}{593767} y_{n+4} - \frac{371569909}{1187534} y_{n+5} = \frac{210h}{593767}[1657999 f_{n+5} + 419450 f_{n+6}] + \frac{900h^2}{593767}[628935 g_{n+4} + 315149 g_{n+5} + 24345 g_{n+6}]$$

$$y_{n+6} = -\frac{4}{41929} y_n + \frac{54}{41929} y_{n+1} - \frac{375}{41929} y_{n+2} + \frac{2000}{41929} y_{n+3} + \frac{13500}{41929} y_{n+4} + \frac{53754}{41929} y_{n+5} - \frac{180h}{41929}[86 f_{n+5} + 89 f_{n+6}] - \frac{1800h^2}{41929}[6 g_{n+5} + g_{n+6}]$$

Order and error constant of the new method

The order and error constants was defined following the method of Chollom *et al.* (2014). The order and error constants of the new method is obtained using

$$[y(t); h] = \sum_{j=0}^{k} \alpha_j y_{n+j} - h\beta_k f_{n+k} - h^2 \gamma_k g_{n+k} \tag{11}$$

The equation can be written in Taylor series expansion about the point x to obtain the expression

$$L[y(x); h] = c_0 y(x) + c_1 h y'^{(x)} + c_2 h^2 y''^{(x)} + \cdots + c_p h^p y^p(x) + \cdots \tag{12}$$

Where the constants $C_p, p = 0, 1,2 \ldots, j = 1,2, \ldots k$ are given as follows:

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_k$$
$$C_1 = (\alpha_0 + 2\alpha_1 + 3\alpha_2 + \cdots + k\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k)$$
$$C_q = \frac{1}{q!}(\alpha_0 + 2^q \alpha_1 + \cdots + k^q \alpha_k) - \frac{1}{(q-1)!}(\beta_1 + 2^{q-1}\beta_2 + \cdots + k^{q-1}\beta_k) - \frac{1}{(q-2)!}(\gamma_1 + 2^{q-2}\gamma_2 + \cdots + k^{q-2}\gamma_k) \tag{13}$$

Hence, Equation (11) is of order p if, $L[y(x); h] = 0(h^{p+2}), C_0 = C_1 = \cdots C_p = C_{P+1} = 0$

$$\text{But } C_{P+2} \neq 0 \tag{14}$$

The truncation error is then given as $C_{P+2} = 10$ in which $C_p = 8$ comparing the coefficient of h gives $C_0 = C_1 = \cdots C_9 = 0 \; and \; C_{10} = \left[-\frac{67181825}{2619860404}, \frac{26730295}{1270871112}, \frac{454165}{9587396}, \frac{7689}{101759}, \frac{5552185}{49876428}, \frac{5}{293503}\right]^T$

Definition: A numerical method is said to be consistent if it is of order greater than one (Lambert, 1991).

Definition: A numerical method is said to be A-stable if the whole of the left-half plane $\{Z: Re(Z) \leq 0\}$ is contained in the region. $\{Z: Re(Z) \leq 1\}$ Where $R(Z)$ is called the stability polynomial of the method (Lambert, 1973)

Definition: A convergence of the new block methods is determined using the approach by Fatunla (1991) and Chollom *et al.* (2007) for linear multistep methods, where the block methods are represented in single block, r point multistep method of the form

$$A^{(0)}y_{m+1} = \sum_{i=1}^{k} A^{(i)}y_{m+1} + h\sum_{i=0}^{k} B^{(i)}f_{m+1} \tag{15}$$

Where h is a fixed mesh size within a block, $A^i, B^i, i = 0,1,2, \dots . k$ are $r \times r$ identity while $y_m, y_{m-1} and\ y_{m+1}$ are vectors of numerical estimates.

The block method can be expressed in the form of (15) gives

$$\begin{pmatrix} -\frac{1120246875}{187132886} & \frac{1904670000}{93566443} & -\frac{5231582500}{93566443} & \frac{16916833125}{93566443} & -\frac{26246727261}{187132886} & 0 \\ \frac{282639375}{121035344} & -\frac{149042750}{7564709} & \frac{436751125}{7564709} & \frac{1356078375}{7564709} & \frac{16690245281}{121035344} & 0 \\ -\frac{1465875}{52678} & \frac{24236500}{1027221} & \frac{38441500}{342407} & -\frac{165139875}{342407} & \frac{766831933}{2054442} & 0 \\ -\frac{4272075}{232592} & \frac{3721350}{14537} & -\frac{5172050}{14537} & -\frac{7541775}{14537} & \frac{147919083}{232592} & 0 \\ -\frac{18295875}{1187534} & \frac{80561000}{593767} & -\frac{758102500}{593767} & \frac{871880625}{593767} & -\frac{371569909}{1187534} & 0 \\ -\frac{54}{41929} & \frac{375}{41929} & -\frac{2000}{41929} & \frac{13500}{41929} & -\frac{53754}{41929} & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{41929} \end{pmatrix} \begin{pmatrix} y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} +$$

$$h \left[\begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{9032409090}{93566443} & -\frac{1362607500}{93566443} \\ 0 & 0 & 0 & 0 & \frac{2871375945}{30258836} & \frac{103377000}{7564709} \\ 0 & 0 & 0 & 0 & \frac{88209590}{342407} & \frac{12211500}{342407} \\ 0 & 0 & 0 & 0 & \frac{26918235}{58148} & \frac{987600}{14537} \\ 0 & 0 & 0 & 0 & \frac{348179790}{593767} & \frac{88084500}{593767} \\ 0 & 0 & 0 & 0 & \frac{15480}{41929} & \frac{16020}{41929} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \end{pmatrix} + \right.$$

$$\left. \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix} \right] + \quad (16)$$

Where

$$A^{(0)} = \begin{pmatrix} -\frac{1120246875}{187132886} & \frac{1904670000}{93566443} & -\frac{5231582500}{93566443} & \frac{16916833125}{93566443} & -\frac{26246727261}{187132886} & 0 \\ \frac{282639375}{121035344} & -\frac{149042750}{7564709} & \frac{436751125}{7564709} & \frac{1356078375}{7564709} & \frac{16690245281}{121035344} & 0 \\ -\frac{1465875}{52678} & \frac{24236500}{1027221} & \frac{38441500}{342407} & -\frac{165139875}{342407} & \frac{766831933}{2054442} & 0 \\ -\frac{4272075}{232592} & \frac{3721350}{14537} & -\frac{5172050}{14537} & -\frac{7541775}{14537} & \frac{147919083}{232592} & 0 \\ -\frac{18295875}{1187534} & \frac{80561000}{593767} & -\frac{758102500}{593767} & \frac{871880625}{593767} & -\frac{371569909}{1187534} & 0 \\ -\frac{54}{41929} & \frac{375}{41929} & -\frac{2000}{41929} & \frac{13500}{41929} & -\frac{53754}{41929} & 1 \end{pmatrix}, A^{(1)} =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{41929} \end{pmatrix}, \ B^{(0)} =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{9032409090}{93566443} & -\frac{1362607500}{93566443} \\ 0 & 0 & 0 & 0 & \frac{2871375945}{30258836} & \frac{103377000}{7564709} \\ 0 & 0 & 0 & 0 & \frac{88209590}{342407} & \frac{12211500}{342407} \\ 0 & 0 & 0 & 0 & \frac{26918235}{58148} & \frac{987600}{14537} \\ 0 & 0 & 0 & 0 & \frac{348179790}{593767} & \frac{88084500}{593767} \\ 0 & 0 & 0 & 0 & \frac{15480}{41929} & \frac{16020}{41929} \end{pmatrix}, \ B^{(1)} =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad C^{(0)} =$$

Substituting $A^{(0)}$ and $A^{(1)}$ into (15) gives the characteristic polynomial of the block method $\rho(\lambda) = \det\,(\lambda A^{(0)} - A^{(1)})$

$$
= det\left[\lambda\begin{pmatrix}
-\frac{1120246875}{187132886} & \frac{1904670000}{93566443} & -\frac{5231582500}{93566443} & \frac{16916833125}{93566443} & -\frac{26246727261}{187132886} & 0 \\
\frac{282639375}{121035344} & -\frac{149042750}{7564709} & \frac{436751125}{7564709} & \frac{1356078375}{7564709} & \frac{16690245281}{121035344} & 0 \\
-\frac{1465875}{52678} & \frac{24236500}{1027221} & \frac{38441500}{342407} & -\frac{165139875}{342407} & \frac{766831933}{2054442} & 0 \\
-\frac{4272075}{232592} & \frac{3721350}{14537} & -\frac{5172050}{14537} & -\frac{7541775}{14537} & \frac{147919083}{232592} & 0 \\
-\frac{18295875}{1187534} & \frac{80561000}{593767} & -\frac{758102500}{593767} & \frac{871880625}{593767} & -\frac{371569909}{1187534} & 0 \\
-\frac{54}{41929} & \frac{375}{41929} & -\frac{2000}{41929} & \frac{13500}{41929} & -\frac{53754}{41929} & 1
\end{pmatrix}
- \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & \frac{4}{41929}
\end{pmatrix}\right]
$$

$$
= det\begin{bmatrix}
-\frac{1120246875}{18713288}\lambda & \frac{1904670000}{93566443}\lambda & -\frac{5231582500}{93566443}\lambda & \frac{16916833125}{93566443}\lambda & -\frac{26246727261}{187132886}\lambda & \lambda \\
\frac{282639375}{121035344}\lambda & -\frac{149042750}{7564709}\lambda & \frac{436751125}{7564709}\lambda & \frac{1356078375}{7564709}\lambda & \frac{16690245281}{121035344}\lambda & \lambda \\
-\frac{1465875}{52678}\lambda & \frac{24236500}{1027221}\lambda & \frac{38441500}{342407}\lambda & -\frac{165139875}{342407}\lambda & \frac{766831933}{2054442}\lambda & \lambda \\
-\frac{4272075}{232592}\lambda & \frac{3721350}{14537}\lambda & -\frac{5172050}{14537}\lambda & -\frac{7541775}{14537}\lambda & \frac{147919083}{232592}\lambda & \lambda \\
-\frac{18295875}{1187534}\lambda & \frac{80561000}{593767}\lambda & -\frac{758102500}{593767}\lambda & \frac{871880625}{593767}\lambda & -\frac{371569909}{1187534}\lambda & \lambda \\
-\frac{54}{41929}\lambda & \frac{375}{41929}\lambda & -\frac{2000}{41929}\lambda & \frac{13500}{41929}\lambda & -\frac{53754}{41929}\lambda & \frac{4192}{4192}
\end{bmatrix}
$$

$$= \lambda^5(\lambda - 1) = 0$$

Therefore, $\lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$.The block method (15) by definition is A-stable and by Henrici (1962) the block method is convergent.

This is obtained with Maple 18 software, hence using Mat lab R2008b software we obtained the region of absolute stability as shown in Figure 1 below.

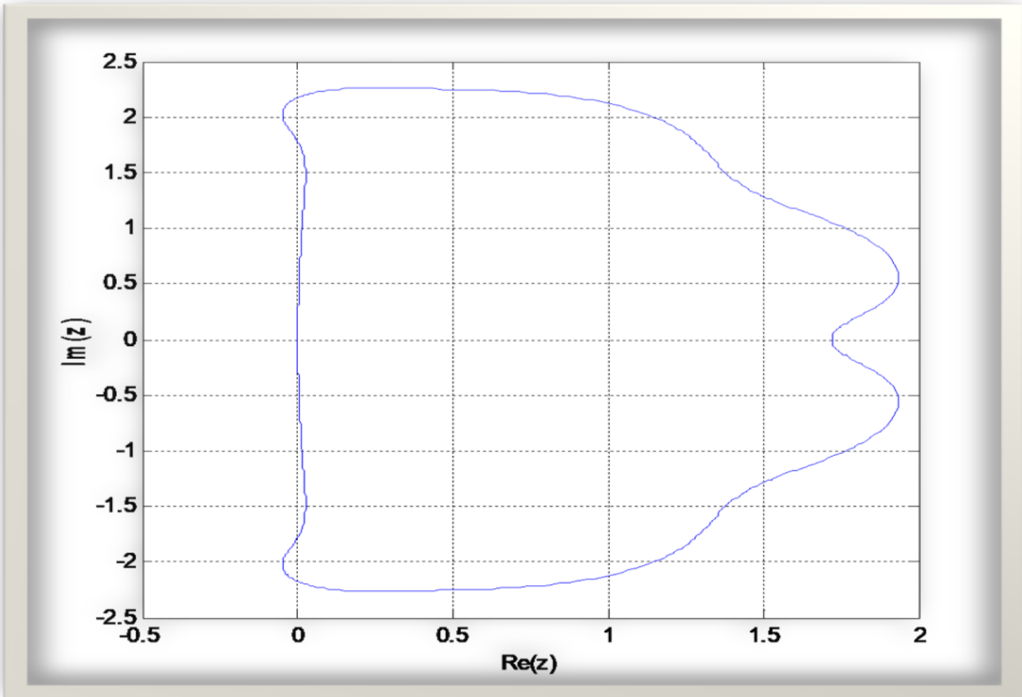

Figure 1: Region of Absolute Stability for the Method $k = 6$

3. NUMERICAL EXPERIMENT

Numerical results on some problems

Example 1:

$y_1' = 0.01y_1 - y_2 + y_3$
$y_2' = y_1 - 100.005y_2 + 99.995y_3$
$y_3' = 2y_1 + 99.995y_2 - 100.005y_3$

Exact $y_{1(x)} = e^{-0.01x}(cos2x + sin2x)$ $y_1(0) = 1$

$y_2(x) = e^{-0.01x}(cos2x + sin2x) + e^{-200x}$ $y_2(0) = 1$

$y_3(x) = e^{-0.01x}(cos2x + sin2x) - e^{-200x}$ $y_3(0) = 1$

$with\ h = 0.1$

Example 2: $y_1' = -12y_1 + 10y_2^2$

$y_2' = y_1 - y_2 - y_1y_2$ $with\ h = 0.1$

Exact $y_{1(x)} = e^{-2x} y_1(0) = 1$

$y_2(x) = e^{-x} y_2(0) = 1$

Example 3: $y_1' = -2.000001y_1 + 0.000001y_2^2$

$y_2' = y_1 - y_2 - y_2^2$ $with\ h = 0.01$

Exact $y_{1(x)} = e^{-2x} y_1(0) = 1$

$y_2(x) = e^{-x} y_2(0) = 1$

Table 1: The Absolute errors of the proposed numerical method solving Example 1 over the interval $\mathbf{0 \leq x \leq 1}$

	ERRORS IN PROPOSED METHOD		
x	$y(x_1)$	$y(x_2)$	$y(x_3)$
0.1	1.21445E-11	9.34180E-10	9.34180E-11
0.2	3.01362E-11	2.89010E-09	2.88901E-10
0.3	8.01496E-11	8.00544E-09	8.00544E-10
0.4	2.18123E-10	2.18656E-08	2.18656E-09
0.5	5.94608E-10	5.96084E-08	5.96084E-09
0.6	1.62060E-09	1.62458E-07	1.62458E-08
0.7	4.41664E-09	4.42752E-07	4.42752E-08
0.8	1.20367E-08	1.20664E-06	1.20664E-07
0.9	3.28039E-08	3.28848E-06	3.28848E-07
1.0	8.94008E-08	8.96212E-06	8.96212E-07

Table 2: The Absolute errors of the proposed numerical method solving Example 2 over the interval $\mathbf{0 \leq x \leq 1}$

	ERRORS IN PROPOSED METHOD	
x	$y(x_1)$	$y(x_2)$
0.1	5.48220E-07	1.39584E-08
0.2	5.08905E-08	7.66810E-09
0.3	1.95076E-09	3.00641E-09
0.4	3.56654E-09	1.06065E-09
0.5	1.71483E-09	3.57053E-10
0.6	6.91775E-10	1.16718E-10
0.7	2.65141E-10	3.71614E-11
0.8	9.98820E-11	1.14713E-11
0.9	3.73978E-11	3.39313E-12
1.0	1.39719E-11	9.38855E-13

Table 3: The Absolute errors of the proposed numerical method solving Example 3 over the interval $\mathbf{0.01 \leq x \leq 0.1}$

	ERRORS IN PROPOSED METHOD	
x	$y(x_1)$	$y(x_2)$
0.01	2.09739E-12	6.58647E-13
0.02	1.60427E-11	2.13817E-12
0.03	1.18629E-10	5.94678E-12
0.04	8.76576E-10	1.62096E-11
0.05	6.47709E-09	4.40775E-11
0.06	4.78596E-08	1.19822E-10
0.07	3.53637E-07	3.25713E-10
0.08	2.61305E-06	8.85387E-10
0.09	1.93079E-05	2.40674E-09
0.10	1.42667E-04	6.54219E-09

Table 4: Comparison of the absolute errors between the proposed method and the method presented in Donald *et al.* (2022) over the interval $\mathbf{0 \leq x \leq 1}$

	ERRORS IN PROPOSED METHOD			ERRORS IN DONALD *ET AL.* (2022)	
x	$y(x_1)$	$y(x_2)$	$y(x_3)$	$y(x_1)$	$y(x_2)$
0.1	1.21445E-11	9.34180E-10	9.34180E-11	3.81739E-05	8.10845E-02
0.2	3.01362E-11	2.89010E-09	2.88901E-10	2.22653E-04	1.88071E-03
0.3	8.01496E-11	8.00544E-09	8.00544E-10	1.51000E-06	1.27320E-05
0.4	2.18123E-10	2.18656E-08	2.18656E-09	7.00000E-08	4.40000E-07
0.5	5.94608E-10	5.96084E-08	5.96084E-09	1.00000E-09	3.00000E-09
0.6	1.62060E-09	1.62458E-07	1.62458E-08	0.00000E+00	0.00000E+00
0.7	4.41664E-09	4.42752E-07	4.42752E-08	1.70000E-09	1.00000E-10
0.8	1.20367E-08	1.20664E-06	1.20664E-07	2.00000E-10	1.70000E-09
0.9	3.28039E-08	3.28848E-06	3.28848E-07	4.00000E-10	1.00000E-10
1.0	8.94008E-08	8.96212E-06	8.96212E-07	3.00000E-10	1.30000E-09

Table 5: Comparison of the absolute errors between the proposed method with the method presented in Mohammed (2011) in Standard Adams Moulton Method over the interval $0 \leq x \leq 1$

	ERROR IN PROPOSED METHOD		ERROR IN MOHAMMED (2011)	ERROR IN STANDARD ADAMS MOULTON METHOD
x	$y(x_1)$	$y(x_2)$		
0.1	5.48220E-07	1.39584E-08	7.9958000E-05	4.1836E-05
0.2	5.08905E-08	7.66810E-09	9.9035000E-05	5.516E-06
0.3	1.95076E-09	3.00641E-09	1.0865900E-04	1.012615E-03
0.4	3.56654E-09	1.06065E-09	1.2054400E-04	2.278645E-03
0.5	1.71483E-09	3.57053E-10	1.3243700E-04	1.522784E-03
0.6	6.91775E-10	1.16718E-10	2.7818900E-04	2.998418E-03
0.7	2.65141E-10	3.71614E-11	3.2503600E-04	1.214683E-03
0.8	9.98820E-11	1.14713E-11	3.5791200E-04	7.120382E-03
0.9	3.73978E-11	3.39313E-12	3.9630900E-04	9.837978E-03
1.0	1.39719E-11	9.38855E-13	4.3703900E-04	1.292385E02

Figure 1 is achieved by applying the developed method to the linear test equation$y' = \lambda y$ and then analyze the resulting iteration to find the set of complex values $Z = \lambda y$ for which the method produces a decaying or stable solution. Table 1, Table 2 and Table 3 the numerical results showed excellent agreement with the exact solution. The absolute errors remained small and well-behaved over the interval, demonstrating the accuracy and reliability of the proposed method for solving the given stiff systems. While Table 4 demonstrates that the constructed method yields better results compared to Donald *et al.* (2022), Mohammed (2011) and Standard Adams Moulton Method.

4. CONCLUSION

In this research, the construct six –step non-hybrid block scheme linear multistep method with continuous coefficients for the approximate solution of first order stiff ordinary differential equation with initial conditions. Three numerical examples have been considered to test the efficiency and accuracy of our methods. This was achieved on account of good stability properties of the new block method and MATLAB codes were being written to test the numerical performance of the block method. The improvement highlights the method's reliability and potential for solving stiff ordinary differential equations more efficient, especially in cases where precision is critical.

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